

The Cartan, Choquet and Kellogg properties for the fine topology on metric spaces

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Abstract. We prove the Cartan and Choquet properties for the fine topology on a complete metric space equipped with a doubling measure supporting a p -Poincaré inequality, $1 < p < \infty$. We apply these key tools to establish a fine version of the Kellogg property, characterize finely continuous functions by means of quasicontinuous functions, and show that capacitary measures associated with Cheeger supersolutions are supported by the fine boundary of the set.

Key words and phrases: Capacitary measure, Cartan property, Choquet property, doubling measure, fine Kellogg property, finely continuous, finely open, fine topology, metric space, nonlinear potential theory, p -harmonic, Poincaré inequality, quasicontinuous, quasiopen, thin.

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1. Introduction

The aim of this paper is to establish the Cartan and Choquet properties for the fine topology on a complete metric space X equipped with a doubling measure μ supporting a p -Poincaré inequality, $1 < p < \infty$. These properties are crucial for deep applications of the fine topology in potential theory. As applications of these key tools we establish the fine Kellogg property and characterize finely continuous functions by means of quasicontinuous functions. We also show that capacitary measures associated with Cheeger p -supersolutions are supported by the fine boundary of the set (not just by the metric boundary).

The classical fine topology is closely related to the Dirichlet problem for the Laplace equation. Wiener [53] showed in 1924 that a boundary point of a domain is irregular if and only if the complement is thin at that point in a certain capacity density sense, cf. Definition 6.1. In 1939 Brelot [22], [23] characterized thinness by a condition, which is nowadays called the Cartan property. The reason for this name is that Cartan (in a letter to Brelot in 1940, see Brelot [24, p. 14]) connected

the notion of thinness to the coarsest topology making all superharmonic functions continuous. Cartan [26] coined the name fine topology for such a topology.

Nonlinear potential theory associated with p -harmonic functions has been studied since the 1960s. For extensive treatises and notes on the history, see the monographs Adams–Hedberg [1], Heinonen–Kilpeläinen–Martio [37] and Malý–Ziemer [49]. Starting in the 1990s a lot of attention has been given to analysis on metric spaces, see e.g. Hajlasz–Koskela [32], Heinonen [35] and Heinonen–Koskela [38]. Around 2000 this initiated studies of p -harmonic and p -superharmonic functions on metric spaces without a differentiable structure, see e.g. Björn–Björn [9], Björn–Björn–Shanmugalingam [15], Kinnunen–Martio [43], Kinnunen–Shanmugalingam [44] and Shanmugalingam [52].

The classical linear fine potential theory and fine topology (the case $p = 2$) have been systematically studied since the 1960s. Let us here just mention Brelot [25], Fuglede [30], [31] and Lukeš–Malý–Zajíček [48], which include most of the theory and the main references. Some of these works are written in large generality including topological spaces, general capacities and families of functions, and some results thus apply also to the nonlinear theory. At the same time, many other results rely indirectly on a linear structure, e.g. through potentials, integral representations and convex cones of superharmonic functions, which are in general not available in the nonlinear setting.

The nonlinear fine potential theory started in the 1970s on unweighted \mathbf{R}^n , see the notes to Chapter 12 in [37] and Section 2.6 in [49]. For the fine potential theory associated with p -harmonic functions on unweighted \mathbf{R}^n , see [49] and Latvala [47]. The monograph [37] is the main source for fine potential theory on weighted \mathbf{R}^n (note that Chapter 21, which is only in the second addition, contains some more recent results). The study of fine potential theory on metric spaces is more recent, see e.g. Björn–Björn [10], Björn–Björn–Latvala [11], J. Björn [20], Kinnunen–Latvala [42] and Korte [45]. For further references to nonlinear and fine nonlinear potential theory, see the introduction to [11].

Recently, in [11], we established the so-called *weak Cartan property*, which says that if $E \subset X$ is thin at $x_0 \notin E$, then there exist a ball $B \ni x_0$ and superharmonic functions u, u' on B such that

$$v(x_0) < \liminf_{E \ni x \rightarrow x_0} v(x),$$

where $v = \max\{u, u'\}$.

The superharmonic functions considered in [11] were based on upper gradients, and because of the lack of a differential equation, we did not succeed in obtaining the full Cartan property as in \mathbf{R}^n , where v itself can be chosen superharmonic, cf. Theorem 1.1 below. Indeed, the proof of the full Cartan property seems to be as hard as the proof of the Wiener criterion, which is also open in the nonlinear potential theory based on upper gradients, but is known to hold in the potential theory based on Cheeger gradients, see J. Björn [19]. Nevertheless, the weak Cartan property in [11] was enough to conclude that the fine topology is the coarsest one making all superharmonic functions continuous.

Here, we instead focus on Cheeger superharmonic functions based on Cheeger's theorem yielding a vector-valued Cheeger gradient. In this case we do have an equation available and this enables us to establish the following full Cartan property.

Theorem 1.1. (Cartan property) *Suppose that E is thin at $x_0 \in \bar{E} \setminus E$. Then there is a bounded positive Cheeger superharmonic function u in an open neighbourhood of x_0 such that*

$$u(x_0) < \liminf_{E \ni x \rightarrow x_0} u(x).$$

For a Newtonian function, the minimal p -weak upper gradient and the modulus of the Cheeger gradient are comparable. Thus the corresponding capacities are comparable to each other, and the fine topology, as well as thinness (and thickness), is the same in both cases. Superminimizers, superharmonic and p -harmonic functions are however different. Hence, using the Cheeger structure we can study thinness and the fine topology, but not e.g. the superharmonic and p -harmonic functions based on upper gradients. Only Cheeger p -(super)harmonic functions can be treated.

We use the Cartan property to establish the following important Choquet property.

Theorem 1.2. (Choquet property) *For any $E \subset X$ and any $\varepsilon > 0$ there is an open set G containing all the points in X at which E is thin, such that $C_p(E \cap G) < \varepsilon$.*

The Choquet property was first established by Choquet [28] in 1959. In the nonlinear theory on unweighted \mathbf{R}^n it was later established by Hedberg [33] and Hedberg–Wolff [34] in connections with potentials (also for higher-order Sobolev spaces). The Cartan property for p -superharmonic functions on unweighted \mathbf{R}^n was obtained by Kilpeläinen–Malý [41] as a consequence of their pointwise Wolff-potential estimates. In fact, Kilpeläinen and Malý used the Cartan property to establish the necessity in the Wiener criterion. In Malý–Ziemer [49], the authors deduce the Choquet property from the Cartan property. The proof of the Cartan property was extended to weighted \mathbf{R}^n by Mikkonen [50, Theorem 5.8] and can also be found in Heinonen–Kilpeläinen–Martio [37, Theorem 21.26 (which is only in the second edition)]; in both places they however refrained from deducing consequences such as the Choquet property.

Our proof of the Choquet property follows the one in [49], but we have some extra complications due to the fact that we can simultaneously have some points with zero capacity and others with positive capacity. Note that Fuglede [30] contains a proof of the Choquet property, in an axiomatic setting, assuming that Corollary 1.3 and part (a) in Theorem 1.4 are true. We have a converse approach, since our proofs of Corollary 1.3 and Theorem 1.4 are based on the Choquet property.

Corollary 1.3. (Fine Kellogg property) *For any $E \subset X$ we have*

$$C_p(\{x \in E : E \text{ is thin at } x\}) = 0. \quad (1.1)$$

The fine Kellogg property has close connections with boundary regularity, see Remark 7.3. The implications \Rightarrow in the following result were already obtained in Björn–Björn–Latvala [11], but now we are able to complete the picture.

Theorem 1.4. (a) *A set $U \subset X$ is quasiopen if and only if $U = V \cup E$ for some finely open set V and for a set E of capacity zero.*

(b) *An extended real-valued function on a quasiopen set U is quasicontinuous in U if and only if u is finite q.e. and finely continuous q.e. in U .*

It is pointed out in Adams–Lewis [2, Proposition 3] that (a) for unweighted \mathbf{R}^n follows from the Choquet property established in Hedberg–Wolff [34]. Also (b) then follows by modifying the earlier axiomatic argumentation of Fuglede [30, Lemma, p. 143]. The proof of Theorem 1.4 in unweighted \mathbf{R}^n is given in Malý–Ziemer [49, p. 146]. For the reader’s convenience, we include the proof of Theorem 1.4 although the proof essentially follows [49]. In Section 8, we use Theorem 1.4 to extend and simplify some recent results from Björn–Björn [10].

We end the paper with another application of the Cartan property in Section 9, which contains results on capacitary measures related to Cheeger supersolutions, see Theorem 9.1 and Corollaries 9.6 and 9.7. In particular, we show that the capacitary measure only charges the fine boundary of the corresponding set.

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2. Notation and preliminaries

We assume throughout the paper that $1 < p < \infty$ and that $X = (X, d, \mu)$ is a metric space equipped with a metric d and a positive complete Borel measure μ such that $0 < \mu(B) < \infty$ for all (open) balls $B \subset X$. It follows that X is separable. The σ -algebra on which μ is defined is obtained by the completion of the Borel σ -algebra. We also assume that $\Omega \subset X$ is a nonempty open set.

We say that μ is *doubling* if there exists a *doubling constant* $C > 0$ such that for all balls $B = B(x_0, r) := \{x \in X : d(x, x_0) < r\}$ in X ,

$$0 < \mu(2B) \leq C\mu(B) < \infty.$$

Here and elsewhere we let $\delta B = B(x_0, \delta r)$. A metric space with a doubling measure is proper (i.e. closed and bounded subsets are compact) if and only if it is complete. See Heinonen [35] for more on doubling measures.

A *curve* is a continuous mapping from an interval, and a *rectifiable* curve is a curve with finite length. We will only consider curves which are nonconstant, compact and rectifiable. A curve can thus be parameterized by its arc length ds . We follow Heinonen and Koskela [38] in introducing upper gradients as follows (they called them very weak gradients).

Definition 2.1. A nonnegative Borel function g on X is an *upper gradient* of an extended real-valued function f on X if for all nonconstant, compact and rectifiable curves $\gamma : [0, l_\gamma] \rightarrow X$,

$$|f(\gamma(0)) - f(\gamma(l_\gamma))| \leq \int_\gamma g \, ds, \quad (2.1)$$

where we follow the convention that the left-hand side is ∞ whenever at least one of the terms therein is infinite. If g is a nonnegative measurable function on X and if (2.1) holds for p -almost every curve (see below), then g is a *p -weak upper gradient* of f .

Here we say that a property holds for *p -almost every curve* if it fails only for a curve family Γ with zero p -modulus, i.e. there exists $0 \leq \rho \in L^p(X)$ such that $\int_\gamma \rho \, ds = \infty$ for every curve $\gamma \in \Gamma$. Note that a p -weak upper gradient need not be a Borel function, it is only required to be measurable. On the other hand, every measurable function g can be modified on a set of measure zero to obtain a Borel function, from which it follows that $\int_\gamma g \, ds$ is defined (with a value in $[0, \infty]$) for p -almost every curve γ . For proofs of these and all other facts in this section we refer to Björn–Björn [9] and Heinonen–Koskela–Shanmugalingam–Tyson [39].

The p -weak upper gradients were introduced in Koskela–MacManus [46]. It was also shown there that if $g \in L^p_{\text{loc}}(X)$ is a p -weak upper gradient of f , then one can find a sequence $\{g_j\}_{j=1}^\infty$ of upper gradients of f such that $g_j - g \rightarrow 0$ in $L^p(X)$. If f has an upper gradient in $L^p_{\text{loc}}(X)$, then it has a *minimal p -weak upper gradient* $g_f \in L^p_{\text{loc}}(X)$ in the sense that for every p -weak upper gradient $g \in L^p_{\text{loc}}(X)$ of f we have $g_f \leq g$ a.e., see Shanmugalingam [52]. The minimal p -weak upper gradient is well defined up to a set of measure zero in the cone of nonnegative functions in $L^p_{\text{loc}}(X)$. Following Shanmugalingam [51], we define a version of Sobolev spaces on the metric measure space X .

Definition 2.2. Let for measurable f ,

$$\|f\|_{N^{1,p}(X)} = \left(\int_X |f|^p d\mu + \inf_g \int_X g^p d\mu \right)^{1/p},$$

where the infimum is taken over all upper gradients of f . The *Newtonian space* on X is

$$N^{1,p}(X) = \{f : \|f\|_{N^{1,p}(X)} < \infty\}.$$

The space $N^{1,p}(X)/\sim$, where $f \sim h$ if and only if $\|f - h\|_{N^{1,p}(X)} = 0$, is a Banach space and a lattice, see Shanmugalingam [51]. In this paper we assume that functions in $N^{1,p}(X)$ are defined everywhere (with values in $\mathbf{R} := [-\infty, \infty]$), not just up to an equivalence class in the corresponding function space. For a measurable set $E \subset X$, the Newtonian space $N^{1,p}(E)$ is defined by considering $(E, d|_E, \mu|_E)$ as a metric space on its own. We say that $f \in N_{\text{loc}}^{1,p}(E)$ if for every $x \in E$ there exists a ball $B_x \ni x$ such that $f \in N^{1,p}(B_x \cap E)$. If $f, h \in N_{\text{loc}}^{1,p}(X)$, then $g_f = g_h$ a.e. in $\{x \in X : f(x) = h(x)\}$, in particular $g_{\min\{f,c\}} = g_f \chi_{\{f < c\}}$ for $c \in \mathbf{R}$.

Definition 2.3. The *Sobolev capacity* of an arbitrary set $E \subset X$ is

$$C_p(E) = \inf_u \|u\|_{N^{1,p}(X)}^p,$$

where the infimum is taken over all $u \in N^{1,p}(X)$ such that $u \geq 1$ on E .

The Sobolev capacity is countably subadditive. We say that a property holds *quasieverywhere* (q.e.) if the set of points for which the property does not hold has Sobolev capacity zero. The Sobolev capacity is the correct gauge for distinguishing between two Newtonian functions. If $u \in N^{1,p}(X)$, then $u \sim v$ if and only if $u = v$ q.e. Moreover, Corollary 3.3 in Shanmugalingam [51] shows that if $u, v \in N^{1,p}(X)$ and $u = v$ a.e., then $u = v$ q.e.

A set $U \subset X$ is *quasiopen* if for every $\varepsilon > 0$ there is an open set $G \subset X$ such that $C_p(G) < \varepsilon$ and $G \cup U$ is open. A function u defined on a set $E \subset X$ is *quasicontinuous* if for every $\varepsilon > 0$ there is an open set $G \subset X$ such that $C_p(G) < \varepsilon$ and $u|_{E \setminus G}$ is finite and continuous. If u is quasicontinuous on a quasiopen set U , then it is easily verified that $\{x \in U : u(x) < a\}$ and $\{x \in U : u(x) > a\}$ are quasiopen for all $a \in \mathbf{R}$, cf. Proposition 3.3 in Björn–Björn–Malý [13].

Definition 2.4. We say that X supports a *p -Poincaré inequality* if there exist constants $C > 0$ and $\lambda \geq 1$ such that for all balls $B \subset X$, all integrable functions f on X and all upper gradients g of f ,

$$\int_B |f - f_B| d\mu \leq C \text{diam}(B) \left(\int_{\lambda B} g^p d\mu \right)^{1/p}, \quad (2.2)$$

where $f_B := \int_B f d\mu / \mu(B)$.

In the definition of Poincaré inequality we can equivalently assume that g is a p -weak upper gradient.

In \mathbf{R}^n equipped with a doubling measure $d\mu = w dx$, where dx denotes Lebesgue measure, the p -Poincaré inequality (2.2) is equivalent to the *p -admissibility* of the weight w in the sense of Heinonen–Kilpeläinen–Martio [37], cf. Corollary 20.9 in [37] and Proposition A.17 in [9].

If X is complete and supports a p -Poincaré inequality and μ is doubling, then Lipschitz functions are dense in $N^{1,p}(X)$, see Shanmugalingam [51]. Moreover, all

functions in $N^{1,p}(X)$ and those in $N^{1,p}(\Omega)$ are quasicontinuous, see Björn–Björn–Shanmugalingam [16]. This means that in the Euclidean setting, $N^{1,p}(\mathbf{R}^n)$ is the refined Sobolev space as defined in Heinonen–Kilpeläinen–Martio [37, p. 96], see Björn–Björn [9, Appendix A.2] for a proof of this fact valid in weighted \mathbf{R}^n . This is the main reason why, unlike in the classical Euclidean setting, we do not need to require the functions competing in the definitions of capacity to be 1 in a neighbourhood of E . For recent related progress on the density of Lipschitz functions see Ambrosio–Colombo–Di Marino [3] and Ambrosio–Gigli–Savaré [4].

In Section 6 the fine topology is defined by means of thin sets, which in turn use the variational capacity cap_p . To be able to define the variational capacity we first need a Newtonian space with zero boundary values. We let, for an arbitrary set $A \subset X$,

$$N_0^{1,p}(A) = \{f|_A : f \in N^{1,p}(X) \text{ and } f = 0 \text{ on } X \setminus A\}.$$

One can replace the assumption “ $f = 0$ on $X \setminus A$ ” with “ $f = 0$ q.e. on $X \setminus A$ ” without changing the obtained space $N_0^{1,p}(A)$. Functions from $N_0^{1,p}(A)$ can be extended by zero in $X \setminus A$ and we will regard them in that sense if needed.

Definition 2.5. The *variational capacity* of $E \subset \Omega$ with respect to Ω is

$$\text{cap}_p(E, \Omega) = \inf_u \int_X g_u^p d\mu,$$

where the infimum is taken over all $u \in N_0^{1,p}(\Omega)$ such that $u \geq 1$ on E .

If $C_p(E) = 0$, then $\text{cap}_p(E, \Omega) = 0$. The converse implication is true if μ is doubling and supports a p -Poincaré inequality.

In Section 9 we will need the following simple lemma. For the reader’s convenience we provide the short proof.

Lemma 2.6. *If $u, v \in N^{1,p}(X)$ are bounded, then $uv \in N^{1,p}(X)$.*

Proof. We can assume that $|u|$ and $|v|$ are bounded by 1. Then $|uv| \leq |u|$ and hence $uv \in L^p(X)$. By the Leibniz rule (Theorem 2.15 in Björn–Björn [9]), $g := |u|g_v + |v|g_u$ is a p -weak upper gradient of uv . As $g \leq g_v + g_u \in L^p(X)$ we see that $uv \in N^{1,p}(X)$. \square

Throughout the paper, the letter C will denote various positive constants whose values may vary even within a line. We also write $A \simeq B$ if $C^{-1}A \leq B \leq CA$.

3. Cheeger gradients

Throughout the rest of the paper, we assume that X is complete and supports a p -Poincaré inequality, and that μ is doubling.

In addition to upper gradients we will also use Cheeger gradients in this paper. Their existence is based on the following deep result of Cheeger.

Theorem 3.1. (Theorem 4.38 in Cheeger [27]) *There exists N and a countable collection (U_α, X^α) of pairwise disjoint measurable sets U_α and Lipschitz “coordinate” functions $X^\alpha: X \rightarrow \mathbf{R}^{k(\alpha)}$, $1 \leq k(\alpha) \leq N$, such that $\mu(X \setminus \bigcup_\alpha U_\alpha) = 0$ and for every Lipschitz function $f: X \rightarrow \mathbf{R}$ there exist unique bounded vector-valued functions $d^\alpha f: U_\alpha \rightarrow \mathbf{R}^{k(\alpha)}$ such that for a.e. $x \in U_\alpha$,*

$$\lim_{r \rightarrow 0} \sup_{y \in B(x,r)} \frac{|f(y) - f(x) - d^\alpha f(x) \cdot (X^\alpha(y) - X^\alpha(x))|}{r} = 0,$$

where \cdot denotes the usual inner product in $\mathbf{R}^{k(\alpha)}$.

Cheeger further shows that for a.e. $x \in U_\alpha$, there is an inner product norm $|\cdot|_x$ on $\mathbf{R}^{k(\alpha)}$ such that for all Lipschitz f ,

$$\frac{1}{C}g_f(x) \leq |d^\alpha f(x)|_x \leq Cg_f(x), \quad (3.1)$$

where C is independent of f and x , see p. 460 in Cheeger [27]. As Lipschitz functions are dense in $N^{1,p}(X)$, the “gradients” $d^\alpha f$ extend uniquely to the whole $N^{1,p}(X)$, by Franchi–Hajlasz–Koskela [29, Theorem 10] or Keith [40]. Moreover, (3.1) holds even for functions in $N^{1,p}(X)$.

From now on we drop α and set

$$Df := d^\alpha f \quad \text{in } U_\alpha.$$

On a metric space there is some freedom in choosing the Cheeger structure. On \mathbf{R}^n we will however always make the natural choice $Df = \nabla f$ and let the inner product norm in (3.1) be the Euclidean norm. Here ∇f denotes the Sobolev gradient from Heinonen–Kilpeläinen–Martio [37], which equals the distributional gradient if the weight on \mathbf{R}^n is a Muckenhoupt A_p weight. In this case, $|Df| = g_f$, by Proposition A.13 in Björn–Björn [9].

4. Supersolutions and superharmonic functions

In the literature on potential theory in metric spaces one usually studies the following (super)minimizers based on upper gradients.

Definition 4.1. A function $u \in N_{\text{loc}}^{1,p}(\Omega)$ is a *(super)minimizer* in Ω if

$$\int_{\{\varphi \neq 0\}} g_u^p d\mu \leq \int_{\{\varphi \neq 0\}} g_{u+\varphi}^p d\mu \quad \text{for all (nonnegative) } \varphi \in \text{Lip}_c(\Omega).$$

A *p-harmonic function* is a continuous minimizer.

Here $\text{Lip}_c(\Omega) = \{\varphi \in \text{Lip}(X) : \text{supp } \varphi \Subset \Omega\}$ and $E \Subset \Omega$ if \bar{E} is a compact subset of Ω .

Minimizers were first studied by Shanmugalingam [52], and superminimizers by Kinnunen–Martio [43]. For various characterizations of minimizers and superminimizers see A. Björn [6]. If u is a superminimizer, then its *lsc-regularization*

$$u^*(x) := \text{ess } \liminf_{y \rightarrow x} u(y) = \lim_{r \rightarrow 0} \text{ess } \inf_{B(x,r)} u \quad (4.1)$$

is also a superminimizer and $u^* = u$ q.e., see [43] or Björn–Björn–Parviainen [14]. If u is a minimizer, then u^* is continuous, (by Kinnunen–Shanmugalingam [44] or Björn–Marola [17]), and thus *p-harmonic*. For further discussion and references on the topics in this section see [9].

In this paper, we consider *Cheeger (super)minimizers* and *Cheeger p-harmonic functions* defined by replacing g_u and $g_{u+\varphi}$ in Definition 4.1 by $|Du|$ and $|D(u+\varphi)|$, respectively, where $|\cdot|$ is the inner product norm in (3.1). Due to the vector structure of the Cheeger gradient one can also make the following definition. (There is no corresponding notion for upper gradients.)

Definition 4.2. A function $u \in N_{\text{loc}}^{1,p}(\Omega)$ is a *(super)solution* in Ω if

$$\int_{\Omega} |Du|^{p-2} Du \cdot D\varphi d\mu \geq 0 \quad \text{for all (nonnegative) } \varphi \in \text{Lip}_c(\Omega),$$

where \cdot is the inner product giving rise to the norm in (3.1).

It can be shown that a function is a (super)solution if and only if it is a Cheeger (super)minimizer, the proof is the same as for Theorem 5.13 in Heinonen–Kilpeläinen–Martio [37]. In weighted \mathbf{R}^n , with the choice $Df = \nabla f$, we have $g_f = |Df| = |\nabla f|$ a.e. which implies that (super)minimizers, Cheeger (super)minimizers and (super)solutions coincide, and are the same as in [37].

Let $G \subset X$ be a nonempty bounded open set with $C_p(X \setminus G) > 0$. We consider the following obstacle problem in G .

Definition 4.3. For $f \in N^{1,p}(G)$ and $\psi : G \rightarrow \overline{\mathbf{R}}$ let

$$\mathcal{K}_{\psi,f}(G) = \{v \in N^{1,p}(G) : v - f \in N_0^{1,p}(G) \text{ and } v \geq \psi \text{ q.e. in } G\}.$$

A function $u \in \mathcal{K}_{\psi,f}(G)$ is a *solution of the $\mathcal{K}_{\psi,f}(G)$ -Cheeger obstacle problem* if

$$\int_G |Du|^p d\mu \leq \int_G |Dv|^p d\mu \quad \text{for all } v \in \mathcal{K}_{\psi,f}(G).$$

A solution to the $\mathcal{K}_{\psi,f}(G)$ -Cheeger obstacle problem is easily seen to be a Cheeger superminimizer (i.e. a supersolution) in G . Conversely, a supersolution u in Ω is a solution of the $\mathcal{K}_{u,u}(G)$ -Cheeger obstacle problem for all open $G \Subset \Omega$ with $C_p(X \setminus G) > 0$.

If $\mathcal{K}_{\psi,f}(G) \neq \emptyset$, then there is a solution u of the $\mathcal{K}_{\psi,f}(G)$ -Cheeger obstacle problem, and this solution is unique up to equivalence in $N^{1,p}(G)$. Moreover, u^* is the unique lsc-regularized solution. Conditions for when $\mathcal{K}_{\psi,f}(G) \neq \emptyset$ can be found in Björn–Björn [10]. If the obstacle ψ is continuous, as a function with values in $[-\infty, \infty)$, then u^* is also continuous. These results were obtained for the upper gradient obstacle problem by Kinnunen–Martio [43], where superharmonic functions based on upper gradients were also introduced. As with most of the results in the metric theory their proofs work verbatim for the Cheeger case considered here. Since most of the theory has been developed in the setting of upper gradients, we will often just refer to the upper gradient equivalents of results for Cheeger (super)minimizers.

For $f \in N^{1,p}(G)$, we let $H_G f$ denote the continuous solution of the $\mathcal{K}_{-\infty,f}(G)$ -Cheeger obstacle problem. This function is Cheeger p -harmonic in G and has the same boundary values (in the Sobolev sense) as f on ∂G , and hence is also called the solution of the (Cheeger) Dirichlet problem with Sobolev boundary values.

Definition 4.4. A function $u : \Omega \rightarrow (-\infty, \infty]$ is *Cheeger superharmonic* in Ω if

- (i) u is lower semicontinuous;
- (ii) u is not identically ∞ in any component of Ω ;
- (iii) for every nonempty open set $\Omega' \Subset \Omega$ with $C_p(X \setminus \Omega') > 0$ and all functions $v \in \text{Lip}(X)$, we have $H_{\Omega'} v \leq u$ in Ω' whenever $v \leq u$ on $\partial\Omega'$.

This definition of Cheeger superharmonicity is equivalent to the one in Heinonen–Kilpeläinen–Martio [37], see A. Björn [5]. A locally bounded Cheeger superharmonic function is a supersolution, and all Cheeger superharmonic functions are lsc-regularized. Conversely, any lsc-regularized supersolution is Cheeger superharmonic. See Kinnunen–Martio [43].

Definition 4.5. The *Cheeger capacitary potential* u_E of a set $E \subset G$ in G is the lsc-regularized solution of the $\mathcal{K}_{\chi_E,0}(G)$ -Cheeger obstacle problem.

The *Cheeger variational capacity* of $E \subset G$ is defined as

$$\text{Ch-cap}_p(E, G) = \int_X |Du_E|^p d\mu = \inf_u \int_X |Du|^p d\mu, \quad (4.2)$$

where the infimum is taken over all $u \in N_0^{1,p}(G)$ such that $u \geq 1$ on E .

By (3.1), we have

$$\text{Ch-cap}_p(E, G) \simeq \text{cap}_p(E, G). \quad (4.3)$$

5. Supersolutions and Radon measures

In this section we assume that Ω is a nonempty bounded open set with $C_p(X \setminus \Omega) > 0$.

It was shown in Björn–MacManus–Shanmugalingam [21, Propositions 3.5 and 3.9] that there is a one-to-one correspondence between supersolutions in Ω and Radon measures in the dual $N_0^{1,p}(\Omega)'$. A *Radon measure* is a positive complete Borel measure which is finite on every compact set.

Proposition 5.1. *For every supersolution u in Ω there is a Radon measure $\nu \in N_0^{1,p}(\Omega)'$ such that for all $\varphi \in N_0^{1,p}(\Omega)$,*

$$Tu(\varphi) := \int_{\Omega} |Du|^{p-2} Du \cdot D\varphi \, d\mu = \int_{\Omega} \varphi \, d\nu, \quad (5.1)$$

where \cdot is the inner product giving rise to the norm in (3.1).

Conversely, if $\nu \in N_0^{1,p}(\Omega)'$ is a Radon measure on Ω then there exists a unique lsc-regularized $u \in N_0^{1,p}(\Omega)$ satisfying $Tu = \nu$ in the sense of (5.1) for all $\varphi \in N_0^{1,p}(\Omega)$. Moreover, u is a nonnegative supersolution in Ω .

Remark 5.2. This result is always false if we drop the assumption $C_p(X \setminus \Omega) > 0$. Indeed, if u is a nonnegative lsc-regularized supersolution in Ω , then u is Cheeger superharmonic in Ω . If $C_p(X \setminus \Omega) = 0$, then u has a Cheeger superharmonic extension to X , by Theorem 6.3 in A. Björn [7] (or Theorem 12.3 in [9]), which must be constant, by Corollary 9.14 in [9]. (Note that if Ω is bounded and $C_p(X \setminus \Omega) = 0$, then also X must be bounded.) On the other hand, there are nonzero Radon measures in $N_0^{1,p}(\Omega)'$, so the existence of a corresponding supersolution fails.

Proof of Proposition 5.1. See [21, Propositions 3.5 and 3.9], where the result was stated under stronger assumptions than here, but the proof of this result is valid under our assumptions. In particular, as $C_p(X \setminus \Omega) > 0$, the coercivity of the map T follows from the Poincaré inequality for $N_0^{1,p}$ (also called the p -Friedrichs' inequality), whose proof can be found e.g. in Corollary 5.54 in [9].

In [21] the uniqueness was shown up to equivalence between supersolutions. The pointwise uniqueness for lsc-regularized supersolutions then follows from (4.1). That u is nonnegative follows from Lemma 5.3 below, as $u \equiv 0$ is the lsc-regularized supersolution corresponding to the zero measure. \square

We need the following comparison principle.

Lemma 5.3. *Let $\nu_1, \nu_2 \in N_0^{1,p}(\Omega)'$ be Radon measures such that $\nu_1 \leq \nu_2$. If $u_1, u_2 \in N_0^{1,p}(\Omega)$ are the corresponding lsc-regularized supersolutions given by Proposition 5.1, then $u_1 \leq u_2$ in Ω .*

Proof. Inserting $\varphi = (u_1 - u_2)_+ \in N_0^{1,p}(\Omega)$ into the equation (5.1) for u_1 and u_2 gives

$$\begin{aligned} 0 &\leq \int_{\Omega} \varphi \, d\nu_2 - \int_{\Omega} \varphi \, d\nu_1 \\ &= \int_{\Omega} (|Du_2|^{p-2} Du_2 - |Du_1|^{p-2} Du_1) \cdot D\varphi \, d\mu \\ &= \int_{\{u_1 > u_2\}} (|Du_2|^{p-2} Du_2 - |Du_1|^{p-2} Du_1) \cdot (Du_1 - Du_2) \, d\mu \\ &\leq \int_{\{u_1 > u_2\}} (|Du_2|^{p-1} |Du_1| + |Du_1|^{p-1} |Du_2| - |Du_1|^p - |Du_2|^p) \, d\mu. \end{aligned} \quad (5.2)$$

The Young inequality shows that

$$\begin{aligned} |Du_2|^{p-1}|Du_1| + |Du_1|^{p-1}|Du_2| &\leq \frac{p-1}{p}|Du_2|^p + \frac{1}{p}|Du_1|^p \\ &\quad + \frac{p-1}{p}|Du_1|^p + \frac{1}{p}|Du_2|^p \\ &= |Du_1|^p + |Du_2|^p. \end{aligned} \quad (5.3)$$

Inserting this into (5.2) shows that equality must hold in (5.2) and (5.3) for a.e. x such that $u_1(x) > u_2(x)$. This implies that $Du_1(x) = k(x)Du_2(x)$ for some $k(x) \geq 0$ (by equality in the last step of (5.2)) and $|Du_1(x)| = |Du_2(x)|$ (by equality in the Young inequality) for a.e. x with $u_1(x) > u_2(x)$. It follows that $Du_1(x) = Du_2(x)$ for a.e. such x and hence $D\varphi = 0$ a.e. in Ω . The Poincaré inequality for $N_0^{1,p}$ (e.g. Corollary 5.54 in [9]) then yields

$$\int_{\Omega} \varphi^p d\mu \leq C_{\Omega} \int_{\Omega} |D\varphi|^p d\mu = 0.$$

Hence $\varphi = 0$ a.e. in Ω , i.e. $u_1 \leq u_2$ a.e. in Ω . As u_1 and u_2 are lsc-regularized, it follows that $u_1 \leq u_2$ everywhere in Ω . \square

Remark 5.4. Note that if u_E is the Cheeger capacitary potential of E in Ω , given by Definition 4.5, then u_E is the lsc-regularized solution of the $\mathcal{K}_{\psi,0}(\Omega)$ -Cheeger obstacle problem, where $\psi = 1$ in E and $\psi = -\infty$ otherwise. Hence, for every $\varphi \in N_0^{1,p}(\Omega \setminus E)$ and every $t > 0$, the function $u_E + t\varphi \in \mathcal{K}_{\psi,0}(\Omega)$ and thus

$$0 \leq \int_{\Omega} (|Du_E + tD\varphi|^p - |Du_E|^p) d\mu.$$

Dividing by t and letting $t \rightarrow 0$ shows that

$$\int_{\Omega} |Du_E|^{p-2} Du_E \cdot D\varphi d\mu \geq 0, \quad (5.4)$$

see (2.8) in Malý–Ziemer [49]. Applying this also to $-\varphi$ shows that equality must hold in (5.4). Consequently, the measure $\nu_E = Tu_E$ satisfies

$$\int_{\Omega} \varphi d\nu_E = 0 \quad \text{for every } \varphi \in N_0^{1,p}(\Omega \setminus E). \quad (5.5)$$

We will need the following lemma when proving the Cartan property (Theorem 1.1). Later, in Theorem 9.1, we will generalize this lemma to quasiopen sets and as a consequence obtain that the measure ν_E is supported on the fine boundary $\partial_p E$; that it is supported on the boundary ∂E is well known.

Lemma 5.5. *Let $E \subset \Omega$ be such that $\text{cap}_p(E, \Omega) < \infty$ and let u_E be the Cheeger capacitary potential of E in Ω , with the corresponding Radon measure $\nu_E = Tu_E$. If $G \subset \Omega$ is open and $v \in N^{1,p}(\Omega)$ is bounded and such that $v = 1$ q.e. in $G \cap E$ then*

$$\int_G v d\nu_E = \int_G u_E d\nu_E. \quad (5.6)$$

In particular, $\nu_E(G) = \int_G u_E d\nu_E$, and if $C_p(G \cap E) = 0$ then $\nu_E(G) = 0$.

Proof. For every $\eta \in \text{Lip}_c(G)$ with $0 \leq \eta \leq 1$ we have $\eta(v - u_E) \in N_0^{1,p}(\Omega \setminus E)$. Thus, (5.5) yields that

$$\int_G \eta(v - u_E) d\nu_E = 0.$$

Since $v - u_E$ and G are bounded, dominated convergence and letting $\eta \nearrow \chi_G$ imply (5.6). For the last part, apply this to $v = 1$ and $v = 0$ respectively. \square

6. Thinness and the fine topology

We now define the fine topological notions which are central in this paper.

Definition 6.1. A set $E \subset X$ is *thin* at $x \in X$ if

$$\int_0^1 \left(\frac{\text{cap}_p(E \cap B(x, r), B(x, 2r))}{\text{cap}_p(B(x, r), B(x, 2r))} \right)^{1/(p-1)} \frac{dr}{r} < \infty. \quad (6.1)$$

A set $U \subset X$ is *finely open* if $X \setminus U$ is thin at each point $x \in U$.

It is easy to see that the finely open sets give rise to a topology, which is called the *fine topology*. Every open set is finely open, but the converse is not true in general.

In the definition of thinness, we make the convention that the integrand is 1 whenever $\text{cap}_p(B(x, r), B(x, 2r)) = 0$. This happens e.g. if $X = B(x, 2r)$, but never if $r < \frac{1}{2} \text{diam } X$. Note that thinness is a local property. Because of (4.3), thinness can equivalently be defined using the Cheeger variational capacity Ch-cap_p .

Definition 6.2. A function $u : U \rightarrow \overline{\mathbf{R}}$, defined on a finely open set U , is *finely continuous* if it is continuous when U is equipped with the fine topology and $\overline{\mathbf{R}}$ with the usual topology.

Since every open set is finely open, the fine topology generated by the finely open sets is finer than the metric topology. In fact, it is the coarsest topology making all (Cheeger) superharmonic functions finely continuous, by J. Björn [20, Theorem 4.4], Korte [45, Theorem 4.3] and Björn–Björn–Latvala [11, Theorem 1.1]. See [9, Section 11.6] and [11] for further discussion on thinness and the fine topology.

7. The Cartan, Choquet and Kellogg properties

We start this section by proving the Cartan property (Theorem 1.1). The proof combines arguments in Kilpeläinen–Malý [41, p. 155] with those in Section 2.1.5 in Malý–Ziemer [49]. As in [41], the pointwise estimate (7.1) is essential here. However, to obtain the estimate $\nu_k(B_j) \leq \text{cap}(E_j, B_{j-1})$, in [41] they use the dual characterization of capacity as the supremum of measures on E_j with potentials bounded by 1. A similar estimate follows also from Theorem 2.45 in [49]. Here we instead use a direct derivation of $\nu_k(B_j) \leq \text{cap}(E_j, B_{j-1})$ based on (5.1), Remark 5.4 and Lemma 5.5.

Proof of Theorem 1.1. By Lemma 4.7 in Björn–Björn–Latvala [11], we may assume that E is open. Let $B_j = B(x_0, r_j)$, $r_j = 2^{-j}$, $E_j = E \cap B_j$ and u_j be the Cheeger capacity potential of E_j with respect to B_{j-1} , $j = 1, 2, \dots$. As E_j is open, we have $u_j = 1$ in E_j . Let $k \geq 1$ be an integer to be specified later, but so large that $\text{diam } B_k < \frac{1}{6} \text{diam } X$, and let $\nu_k = Tu_k$ be the Radon measure in $N_0^{1,p}(B_{k-1})'$, given by Proposition 5.1.

Since $u_k = 1$ in E_k , it remains to show that $u_k(x_0) < 1$ for some k . By Remark 5.4 in Kinnunen–Martio [43] (or Proposition 8.24 in [9]), x_0 is a Lebesgue point of u_k . Hence, Proposition 4.10 in Björn–MacManus–Shanmugalingam [21] shows that

$$u_k(x_0) \leq c \left(\int_{B_k} u_k^p d\mu \right)^{1/p} + c \sum_{j=k-1}^{\infty} \left(r_j^p \frac{\nu_k(B_j)}{\mu(B_j)} \right)^{1/(p-1)}. \quad (7.1)$$

The first term in the right-hand side can be estimated using the Sobolev inequality [9, Theorem 5.51] and the fact that $\text{cap}_p(B_k, B_{k-1}) \simeq r_k^{-p} \mu(B_k)$ (by Lemma 3.3 in J. Björn [18] or Proposition 6.16 in [9]) as

$$\int_{B_k} u_k^p d\mu \leq \frac{1}{\mu(B_k)} \int_{B_{k-1}} u_k^p d\mu \leq \frac{Cr_k^p}{\mu(B_k)} \int_{B_{k-1}} |Du_k|^p d\mu \simeq \frac{\text{cap}_p(E_k, B_{k-1})}{\text{cap}_p(B_k, B_{k-1})}. \quad (7.2)$$

Here we have also used (4.2) and (4.3).

As for the second term in (7.1), let v_j be the lsc-regularized solution of $Tv_j = \nu_k|_{B_j}$ in B_{k-1} , $j \geq k$. Lemma 5.3 shows that $v_j \leq u_k \leq 1$ in B_{k-1} . Thus, with v_j as a test function in (5.1), we have

$$\int_{B_{k-1}} |Dv_j|^p d\mu = \int_{B_j} v_j d\nu_k \leq \int_{B_j} u_k d\nu_k. \quad (7.3)$$

Using Lemma 5.5 (for the first equality below) and (5.1) with u_j as a test function (for the third equality) we obtain that

$$\begin{aligned} \int_{B_j} u_k d\nu_k &= \int_{B_j} u_j d\nu_k = \int_{B_{k-1}} u_j d\nu_k|_{B_j} = \int_{B_{k-1}} |Dv_j|^{p-2} Dv_j \cdot Du_j d\mu \\ &\leq \left(\int_{B_{k-1}} |Dv_j|^p d\mu \right)^{1-1/p} \left(\int_{B_{k-1}} |Du_j|^p d\mu \right)^{1/p}. \end{aligned} \quad (7.4)$$

Together with (7.3) this implies that

$$\int_{B_{k-1}} |Dv_j|^p d\mu \leq \int_{B_{k-1}} |Du_j|^p d\mu = \text{Ch-cap}_p(E_j, B_{j-1}),$$

where Ch-cap_p denotes the Cheeger variational capacity. Inserting this into (7.4) yields,

$$\int_{B_j} u_k d\nu_k \leq \text{Ch-cap}_p(E_j, B_{j-1}),$$

which together with the last part of Lemma 5.5 and (4.3) shows that

$$\nu_k(B_j) = \int_{B_j} u_k d\nu_k \leq \text{Ch-cap}_p(E_j, B_{j-1}) \simeq \text{cap}_p(E_j, B_{j-1}).$$

Hence using $\text{cap}_p(B_j, B_{j-1}) \simeq r_j^{-p} \mu(B_j)$ again we obtain

$$\sum_{j=k-1}^{\infty} \left(r_j^p \frac{\nu_k(B_j)}{\mu(B_j)} \right)^{1/(p-1)} \leq C \sum_{j=k-1}^{\infty} \left(\frac{\text{cap}_p(E_j, B_{j-1})}{\text{cap}_p(B_j, B_{j-1})} \right)^{1/(p-1)}. \quad (7.5)$$

Since E is thin at x_0 , both (7.2) and (7.5) can be made arbitrarily small by choosing k large enough. Thus $u_k(x_0) < 1$ for large enough k . \square

We now turn to the proof of the Choquet property (Theorem 1.2). The following notation is common in the literature. The *base* $b_p E$ of a set $E \subset X$ consists of all points $x \in X$ where E is *thick*, i.e. not thin, at x . Using this notation, the Choquet property can be formulated as follows.

Theorem 7.1. (Choquet property) *For any $E \subset X$ and any $\varepsilon > 0$ there is an open set G so that*

$$G \cup b_p E = X \quad \text{and} \quad C_p(E \cap G) < \varepsilon.$$

Proof. Let $\{B_j\}_{j=1}^\infty$ be a countable covering of X by balls such that every point is covered by arbitrarily small balls. Such a covering exists as X is separable. Choose $\varepsilon > 0$. For each j , let u_j be the Cheeger capacitary potential of $E \cap B_j$ with respect to $2B_j$. Since each u_j is quasicontinuous, there is an open set G'_j with $C_p(G'_j) < 2^{-j}\varepsilon$ such that the set

$$G_j := \{x \in B_j : u_j(x) < 1\} \cup G'_j \quad (7.6)$$

is open. We set $G := \bigcup_{j=1}^\infty G_j$ and will show that $G \cup b_p E = X$.

Choose a point $z \in X \setminus b_p E$. If $\text{dist}(z, E \setminus \{z\}) > 0$, then there is $B_j \ni z$ such that $B_j \cap E$ is either empty or $\{z\}$. If $B_j \cap E = \emptyset$, then $u_j \equiv 0$. If $B_j \cap E = \{z\}$, then the thinness of E at z together with Proposition 1.3 in Björn–Björn–Latvala [11] shows that $C_p(\{z\}) = 0$, and hence $u_j \equiv 0$ as well. In both cases we obtain $z \in B_j \subset G_j \subset G$.

We can therefore assume that $z \in \overline{E \setminus \{z\}}$. By Theorem 1.1 (applied to $E \setminus \{z\}$), there is a bounded positive Cheeger superharmonic function v in an open neighbourhood of z such that

$$v(z) < 1 < \liminf_{E \ni x \rightarrow z} v(x).$$

Hence we may fix a ball $B_j \ni z$ so that v is Cheeger superharmonic in $3B_j$ and $v \geq 1$ in $B_j \cap E$. Since v is the lsc-regularized solution of the $\mathcal{K}_{v,v}(2B_j)$ -Cheeger obstacle problem and u_j is the lsc-regularized solution of the $\mathcal{K}_{\chi_{B_j \cap E}, 0}(2B_j)$ -Cheeger obstacle problem, the comparison principle in Lemma 5.4 in Björn–Björn [8] (or Lemma 8.30 in [9]) yields $u_j \leq v$ in $2B_j$. It follows that $u_j(z) < 1$, and thus $z \in G_j \subset G$.

It remains to prove that $C_p(E \cap G) < \varepsilon$. For any j , we have $u_j \geq 1$ q.e. in $E \cap B_j$, and thus (7.6) implies

$$C_p(E \cap G_j) \leq C_p(\{x \in E \cap B_j : u_j(x) < 1\}) + C_p(G'_j) = C_p(G'_j) < 2^{-j}\varepsilon.$$

By the countable subadditivity of the capacity we obtain $C_p(E \cap G) < \varepsilon$. \square

As a consequence of the Choquet property we can now deduce Corollary 1.3. Because of Remark 7.3 below, we find the name fine Kellogg property natural.

Corollary 7.2. (Fine Kellogg property) *For any $E \subset X$ we have*

$$C_p(E \setminus b_p E) = 0.$$

Proof. For every $\varepsilon > 0$, Theorem 7.1 provides us with an open set G such that $G \cup b_p E = X$ and $C_p(E \cap G) < \varepsilon$. Then $E \setminus b_p E \subset E \cap G$, and therefore $C_p(E \setminus b_p E) < \varepsilon$. Letting $\varepsilon \rightarrow 0$ concludes the proof. \square

Remark 7.3. Let $\Omega \subset X$ be a bounded open set with $C_p(X \setminus \Omega) > 0$. Choosing $E = X \setminus \Omega$ in Corollary 7.2 gives

$$C_p(\partial\Omega \setminus b_p(X \setminus \Omega)) \leq C_p((X \setminus \Omega) \setminus b_p(X \setminus \Omega)) = 0. \quad (7.7)$$

On the other hand, a boundary point $x_0 \in \partial\Omega$ is regular (both for p -harmonic functions defined through upper gradients and for Cheeger p -harmonic functions) whenever $X \setminus \Omega$ is thick at x_0 , by the sufficiency part of the Wiener criterion, see Björn–MacManus–Shanmugalingam [21] and J. Björn [19], [20] (or Theorem 11.24 in [9]). Hence (7.7) yields that the set of irregular boundary points of Ω is of capacity zero. This result was obtained by a different method (and called the Kellogg property) in Björn–Björn–Shanmugalingam [15, Theorem 3.9].

To clarify that the above proof of the Kellogg property is not using circular reasoning let us explain how the results we use here are obtained in [9]. Here we only need results up to Chapter 9 therein plus the results in Sections 11.4 and 11.6. They in turn only rely on results up to Chapter 9 plus the implication (b) \Rightarrow (a) in Theorem 10.29, which can easily be obtained just using comparison. Hence we are not relying on the Kellogg property obtained in Section 10.2 in [9].

8. Finely open and quasiopen sets

We start this section by using the Choquet property to prove Theorem 1.4, i.e. we characterize quasiopen sets and quasicontinuity by means of the corresponding fine topological notions. We then proceed by giving several immediate applications of this characterization.

Note that if $C_p(\{x\}) = 0$, then $\{x\}$ is quasiopen, but not finely open. Thus the zero capacity set in Theorem 1.4 (a) cannot be dropped.

Proof of Theorem 1.4. (a) That each quasiopen set U is of the form $U = V \cup E$ for some finely open set V and for a set E of capacity zero, was recently shown in Björn–Björn–Latvala [11, Theorem 4.9].

For the converse, assume that $U = V \cup E$, where V is finely open and $C_p(E) = 0$. Let $\varepsilon > 0$. By the Choquet property (Theorem 7.1), applied to $X \setminus V$, there is an open set G such that

$$G \cup b_p(X \setminus V) = X \quad \text{and} \quad C_p(G \setminus V) < \varepsilon.$$

The capacity C_p is an outer capacity, by Corollary 1.3 in Björn–Björn–Shanmugalingam [16] (or Theorem 5.31 in [9]), so there is an open set $\tilde{G} \supset (G \setminus V) \cup E$ such that $C_p(\tilde{G}) < \varepsilon$. Since V is finely open, we have $V \subset X \setminus b_p(X \setminus V) \subset G$, and thus $U \cup \tilde{G} = V \cup \tilde{G} = G \cup \tilde{G}$ is open, i.e. U is quasiopen.

(b) If u is quasicontinuous, then it is finite q.e., by definition, and finely continuous q.e., by Theorem 4.9 in Björn–Björn–Latvala [11].

Conversely, assume that there is a set Z with $C_p(Z) = 0$ such that u is finite and finely continuous on $V := U \setminus Z$. By (a), we can assume that V is finely open. Let $\varepsilon > 0$ and let $\{(a_j, b_j)\}_{j=1}^\infty$ be an enumeration of all open intervals with rational endpoints and set

$$V_j := \{x \in V : a_j < u(x) < b_j\}.$$

By the fine continuity of u , the sets V_j are finely open. Hence by (a), V_j are quasiopen, and thus there are open sets G_j and G_U with $C_p(G_j) < 2^{-j}\varepsilon$ and $C_p(G_U) < \varepsilon$ such that $V_j \cup G_j$ and $U \cup G_U$ are open. Also, as C_p is an outer capacity, there is an open set $G_Z \supset Z$ with $C_p(G_Z) < \varepsilon$. Then

$$G := G_Z \cup G_U \cup \bigcup_{j=1}^{\infty} G_j$$

is open, $C_p(G) < 3\varepsilon$, and $u|_{U \setminus G}$ is continuous since $V_j \cup G$ are open sets. \square

Theorem 1.4 leads directly to the following improvements of the results in Björn–Björn [10].

Corollary 8.1. *Every finely open set is quasiopen, measurable and p -path open.*

A set U is p -path open if for p -almost every curve $\gamma : [0, l_\gamma] \rightarrow X$, the set $\gamma^{-1}(U)$ is (relatively) open in $[0, l_\gamma]$.

Proof. By Theorem 1.4(a) every finely open set is quasiopen. Hence the result follows from Remark 3.5 in Shanmugalingam [52] and Lemma 9.3 in [10]. \square

An important consequence is that the restriction of a minimal p -weak upper gradient to a finely open set remains minimal. This was shown for measurable p -path open sets in [10, Corollary 3.7]. We restate this result, in view of Corollary 8.1. In order to do so in full generality, we need to introduce some more notation.

We define the Dirichlet space

$$D^p(X) = \{u : u \text{ is measurable and } u \text{ has an upper gradient in } L^p(X)\}.$$

As with $N^{1,p}(X)$ we assume that functions in $D^p(X)$ are defined everywhere (with values in $\overline{\mathbf{R}} := [-\infty, \infty]$). For a measurable set $E \subset X$, the spaces $D^p(E)$ and $D_{\text{loc}}^p(E)$ are defined similarly. For $u \in D_{\text{loc}}^p(E)$ we denote the minimal p -weak upper gradient of u taken with E as the underlying space by $g_{u,E}$. Its existence is guaranteed by Theorem 2.25 in [9].

Corollary 8.2. *Let U be quasiopen and $u \in D_{\text{loc}}^p(X)$. Then $g_{u,U} = g_u$ a.e. in U . In particular this holds if U is finely open.*

Proof. By Remark 3.5 in Shanmugalingam [52] and Lemma 9.3 in [10] every quasiopen set is p -path open and measurable, whereas Theorem 1.4(a) shows that every finely open set is quasiopen. Hence the result follows from Corollary 3.7 in [10]. \square

In [10], the fine topology turned out to be important for obstacle problems on nonopen measurable sets, i.e. when minimizing the p -energy integral

$$\int_E g_{u,E} d\mu \quad (8.1)$$

on an arbitrary bounded measurable set E among all functions

$$u \in \mathcal{K}_{\psi_1, \psi_2, f}(E) := \{v \in D^p(E) : v - f \in N_0^{1,p}(E) \text{ and } \psi_1 \leq v \leq \psi_2 \text{ q.e. in } E\}.$$

Knowing that finely open sets are measurable and p -path open, we are now able to improve and simplify some of the results therein. We summarize these improvements in the following theorem, which follows directly from [10, Theorems 1.2 and 8.3, and Corollaries 3.7 and 7.4] and Corollary 8.1. We denote the fine interior of E by $\text{fine-int } E$.

Theorem 8.3. *Let $E \subset X$ be a bounded measurable set such that $C_p(X \setminus E) > 0$, and let $f \in D^p(E)$ and $\psi_j : E \rightarrow \overline{\mathbf{R}}$, $j = 1, 2$, be such that $\mathcal{K}_{\psi_1, \psi_2, f}(E) \neq \emptyset$. Also let $E_0 = \text{fine-int } E$.*

Then $\mathcal{K}_{\psi_1, \psi_2, f}(E) = \mathcal{K}_{\psi_1, \psi_2, f}(E_0)$, and the solutions of the minimization problem for (8.1) with respect to $\mathcal{K}_{\psi_1, \psi_2, f}(E)$ and $\mathcal{K}_{\psi_1, \psi_2, f}(E_0)$ coincide. Moreover, $g_{u, E_0} = g_{u, E}$ a.e. in E_0 and if $\mu(E \setminus E_0) = 0$ then also the p -energies associated with these two minimization problems coincide.

Furthermore, if $f \in D^p(\Omega)$ for some open set $\Omega \supset E$, then $g_{u, E_0} = g_{u, E} = g_u$ a.e. in E_0 and the above solutions coincide with the solutions of the corresponding $\mathcal{K}_{\psi'_1, \psi'_2, f}(\Omega)$ -obstacle problem, where ψ'_j is the extension of ψ_j to $\Omega \setminus E$ by f , $j = 1, 2$.

We also obtain the following consequence of Lemma 3.9 and Theorem 7.3 in [10], which generalizes Theorem 2.147 and Corollary 2.162 in Malý–Ziemer [49] to metric spaces and to arbitrary sets. See also Remark 2.148 in [49] for another description of $W_0^{1,p}(\Omega)$ in \mathbf{R}^n .

Proposition 8.4. (Cf. Proposition 9.4 in [10].) *Let $E \subset X$ be arbitrary and $u \in N^{1,p}(\overline{E}^p)$, where \overline{E}^p is the fine closure of E . Then $u \in N_0^{1,p}(E)$ if and only if $u = 0$ q.e. on the fine boundary $\partial_p E := \overline{E}^p \setminus \text{fine-int } E$ of E .*

9. Support of capacitary measures

We can now bootstrap Lemma 5.5 to quasiopen sets and in particular show that the capacitary measure ν_E only charges the fine boundary $\partial_p E := \bar{E}^p \setminus \text{fine-int } E$ of E , where \bar{E}^p is the fine closure of E . This observation seems to be new even in unweighted \mathbf{R}^n .

Theorem 9.1. *Let Ω be a nonempty bounded open set with $C_p(X \setminus \Omega) > 0$. Let $E \subset \Omega$, u_E and $\nu_E = Tu_E$ be as in Lemma 5.5. Let $U \subset \Omega$ be quasiopen and $v \in N^{1,p}(\Omega)$. Then the following are true:*

- (a) *If $u \in N^{1,p}(\Omega)$ and either u is bounded from below or belongs to $L^1(\nu_E)$, and $u = v$ q.e. in $U \cap E$, then*

$$\int_U u \, d\nu_E = \int_U v \, d\nu_E. \quad (9.1)$$

- (b) *If $v = 1$ q.e. in $U \cap E$, then*

$$\nu_E(U) = \int_U v \, d\nu_E = \int_U u_E \, d\nu_E.$$

- (c) *If $C_p(U \cap E) = 0$, then $\nu_E(U) = 0$.*

Remark 9.2. We shall see in Corollary 9.6 below that the set $U \cap E$ in (a), (b) and (c) above can be replaced by $U \cap \partial_p E$ and that the assumption $v \in N^{1,p}(\Omega)$ in Theorem 9.1 can be omitted in that case.

To prove Theorem 9.1 we need the following quasi-Lindelöf principle, whose proof in unweighted \mathbf{R}^n is given in Theorem 2.3 in Heinonen–Kilpeläinen–Malý [36]. This proof, which relies on the fine Kellogg property, extends to metric spaces, see Björn–Björn–Latvala [12].

Theorem 9.3. (Quasi-Lindelöf principle) *For each family \mathcal{V} of finely open sets there is a countable subfamily \mathcal{V}' such that*

$$C_p\left(\bigcup_{V \in \mathcal{V}} V \setminus \bigcup_{V' \in \mathcal{V}'} V'\right) = 0.$$

We also need the following lemmas.

Lemma 9.4. *Let U be finely open and let $x_0 \in U$. Then there exists a finely open set $V \Subset U$ containing x_0 and a function $v \in N_0^{1,p}(U)$ such that $v = 1$ on V and $0 \leq v \leq 1$ everywhere.*

Proof. Since U is finely open, $E := X \setminus U$ is thin at x_0 . By the Cartan property (Theorem 1.1), there are a ball $B \ni x_0$ and a lower semicontinuous finely continuous $u \in N^{1,p}(B)$ such that $0 \leq u \leq 1$ in B , $u(x_0) < 1$ and $u = 1$ in $E \cap B$. Let $\eta \in \text{Lip}_c(B)$ be such that $0 \leq \eta \leq 1$ in B and $\eta = 1$ in $\frac{1}{2}B$. Then $w := \eta(1-u) \in N_0^{1,p}(U)$ is upper semicontinuous and finely continuous in X and $w(x_0) = 1 - u(x_0) > 0$. Let $v = \min\{1, 2w/w(x_0)\} \in N_0^{1,p}(U)$ and $V = \{x \in U : w(x) > \frac{1}{2}w(x_0)\}$. The fine continuity and upper semicontinuity of w imply that V is finely open and $V \Subset U$. Moreover $x_0 \in V$ and $v = 1$ on V . \square

Lemma 9.5. *Let $U \subset X$ be quasiopen. Then*

$$U = W_1 \cup E_1 = W_2 \setminus E_2, \quad (9.2)$$

where W_1 and W_2 are Borel sets and E_1 and E_2 are of capacity zero. Moreover, we may choose W_1 to be of type F_σ and W_2 to be of type G_δ .

Not all finely open sets are Borel. Let for instance $V = G \setminus A$, where G is open and $A \subset G$ is a non-Borel set with $C_p(A) = 0$. Then V is a non-Borel finely open set. To be more specific, we may let $A \subset G \subset \mathbf{R}^n$ be any non-Borel set of Hausdorff dimension $< n - p$.

Proof. By definition, for each $j = 1, 2, \dots$ there is an open set G_j such that $U \cup G_j$ is open and $C_p(G_j) < 1/j$. Then

$$\begin{aligned} U &= \left(U \setminus \bigcap_{j=1}^{\infty} G_j \right) \cup \left(U \cap \bigcap_{j=1}^{\infty} G_j \right) = \bigcup_{j=1}^{\infty} (U \setminus G_j) \cup \bigcap_{j=1}^{\infty} (U \cap G_j) \\ &= \bigcup_{j=1}^{\infty} ((U \cup G_j) \setminus G_j) \cup \bigcap_{j=1}^{\infty} (U \cap G_j) =: W_1 \cup E_1. \end{aligned}$$

The second equality in (9.2) follows by choosing $W_2 = \bigcap_{j=1}^{\infty} (U \cup G_j)$ and $E_2 = W_2 \setminus U$. The last two claims follow from the choices above. \square

Proof of Theorem 9.1. By Theorem 1.4, we can find a finely open set $V \subset U$ such that $C_p(U \setminus V) = 0$. For every $x \in V$, Lemma 9.4 provides us with a finely open set $V_x \subseteq V$ containing x and a function $v_x \in N_0^{1,p}(V)$ such that $v_x = 1$ on V_x and $0 \leq v_x \leq 1$ everywhere. By the quasi-Lindelöf principle, and the fact that $C_p(U \setminus V) = 0$, we can out of these choose $V_j = V_{x_j}$ and $v_j = v_{x_j}$, $j = 1, 2, \dots$, so that $U = \bigcup_{j=1}^{\infty} V_j \cup Z$, where $C_p(Z) = 0$. For $k = 1, 2, \dots$, set

$$\eta_k = \chi_{X \setminus Z} \max_{j=1,2,\dots,k} v_j \in N_0^{1,p}(U).$$

Since ν_E is a complete Borel measure which, by Lemma 5.5 (or Lemma 3.8 in Björn–MacManus–Shanmugalingam [21]), is absolutely continuous with respect to the capacity C_p , it follows from Lemma 9.5 that U is ν_E -measurable and $\nu_E(Z) = 0$. We are now ready to prove (a)–(c).

(a) First, assume that u and v are bounded. Then $\eta_k(u - v) \in N^{1,p}(U)$, by Lemma 2.6, and Lemma 2.37 in Björn–Björn [9] shows that $\eta_k(u - v) \in N_0^{1,p}(U)$. Since $u = v$ q.e. in $U \cap E$, it follows that $\eta_k(u - v) \in N_0^{1,p}(U \setminus E)$. Hence (5.5) yields that

$$\int_U \eta_k(u - v) d\nu_E = 0.$$

Since $\eta_k \nearrow \chi_{U \setminus Z}$ in U , dominated convergence and the fact that $\nu_E(Z) = 0$ imply that

$$\int_U (u - v) d\nu_E = \int_{U \setminus Z} (u - v) d\nu_E = 0,$$

and (9.1) follows.

Next, assume that u and v are bounded from below. Then, by monotone convergence and the bounded case,

$$\int_U u d\nu_E = \lim_{k \rightarrow \infty} \int_U \min\{u, k\} d\nu_E = \lim_{k \rightarrow \infty} \int_U \min\{v, k\} d\nu_E = \int_U v d\nu_E.$$

Finally, applying this to the positive and negative parts of u and v gives

$$\int_U u_+ d\nu_E = \int_U v_+ d\nu_E \quad \text{and} \quad \int_U u_- d\nu_E = \int_U v_- d\nu_E,$$

and hence

$$\int_U u d\nu_E = \int_U u_+ d\nu_E - \int_U u_- d\nu_E = \int_U v_+ d\nu_E - \int_U v_- d\nu_E = \int_U v d\nu_E,$$

where the assumptions on u guarantee that the subtractions are well defined (i.e. not $\infty - \infty$).

(b) By applying (a) to $u = u_E$ and v we have $\int_U v d\nu_E = \int_U u_E d\nu_E$. Choosing $v \equiv 1$ yields $\nu_E(U) = \int_U u_E d\nu_E$.

(c) This follows by applying (b) to $v \equiv 0$. \square

Corollary 9.6. *Let Ω , E , u_E and ν_E be as in Theorem 9.1. Then*

$$\nu_E(\Omega \setminus \partial_p E) = 0,$$

i.e. ν_E is supported on the fine boundary $\partial_p E := \overline{E}^p \setminus \text{fine-int } E$ of E .

Proof. First, the fine exterior $V = \Omega \setminus \overline{E}^p$ is finely open and $V \cap E = \emptyset$, whence $\nu_E(V) = 0$ by Theorem 9.1 (c).

Next, the fine interior $E_0 := \text{fine-int } E$ is finely open and as in the proof of Theorem 9.1 we can use the quasi-Lindelöf principle to find nonnegative $\eta_k \in N_0^{1,p}(E_0)$ such that

$$\eta_k \nearrow \chi_{E_0 \setminus Z} \quad \text{as } k \rightarrow \infty,$$

where $C_p(Z) = 0$. Since $u_E = 1$ q.e. in E , we have $Du_E = 0$ a.e. in E and hence by (5.1)

$$\int_{\Omega} \eta_k d\nu_E = \int_{\Omega} |Du_E|^{p-2} Du_E \cdot D\eta_k d\mu = 0.$$

Dominated convergence then shows that $\nu_E(E_0 \setminus Z) = 0$. Since $\nu_E(Z) = 0$ by Lemma 5.5 (or Lemma 3.8 in [21]), the proof is complete. \square

Corollary 9.7. *Let Ω , $E \subset \Omega$, u_E and $\nu_E = Tu_E$ be as in Lemma 5.5. Let $U \subset \Omega$ be quasiopen. Then the following are true:*

(a) *If u is a function on Ω such that $\int_{U \cap \partial_p E} u d\nu_E$ is well-defined and v is a function on U such that $v = u$ q.e. in $U \cap \partial_p E$, then*

$$\int_U v d\nu_E = \int_U u d\nu_E.$$

(b) *If $v = 1$ q.e. in $U \cap \partial_p E$, then*

$$\nu_E(U) = \int_U v d\nu_E = \int_U u_E d\nu_E.$$

(c) *If $C_p(U \cap \partial_p E) = 0$, then $\nu_E(U) = 0$.*

Proof. (c) This follows directly from Corollary 9.6 and the fact that ν_E is absolutely continuous with respect to the capacity C_p (by Lemma 5.5).

(a) By Corollary 9.6 and the absolute continuity of ν_E with respect to C_p again, we see that

$$\int_U v d\nu_E = \int_{U \cap \partial_p E} v d\nu_E = \int_{U \cap \partial_p E} u d\nu_E = \int_U u d\nu_E.$$

(b) This follows from (a) by choosing $u \equiv 1$ and $u = u_E$, respectively. \square

We end with a simple example showing that the fine boundary can be much smaller than the metric boundary. A much more involved example in the same spirit is given in Section 9 in Björn–Björn [10].

Example 9.8. Let B be an open ball in \mathbf{R}^n , $1 < p \leq n$, and let $E = B \setminus \mathbf{Q}^n$. The set E is finely open and has fine closure $\overline{E}^p = \overline{B}$. Hence $\partial_p E = \partial B \cup (B \cap \mathbf{Q}^n)$, while $\partial E = \overline{B}$.

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